Maximally Entangled States via Mutual Unbiased Collective Bases

M. Revzen[1]

[1]*Department of Physics, Technion - Israel Institute of Technology, Haifa 32000, Israel*
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Relative and center of mass coordinates are used to generalize mutually unbiased bases (MUB) and define mutually unbiased collective bases (MUCB). Maximal entangled states are given as product states in the collective variables.

INTRODUCTION

Entanglement is central to some of the intriguing counter intuitive aspects of quantum mechanics, yet, no accepted entanglement measure, applicable for both pure and mixed states, is available. Nontheless for two particles (systems) of equal (Hilbert space) dimensionality, d, it is intuitively clear that a pure state, $\psi(1,2)$, whose Schmidt's decomposition [1] is given by,

$$\psi(1,2) = \frac{1}{\sqrt{d}}\Sigma_{n=1}^{d} a_n u_n(1) v_n(2), \tag{1}$$

with $|a_n| = 1$ and $\langle u_n|u_m\rangle = \langle v_n|v_m\rangle = \delta_{n,m}$, is a *maximally entangled* state (MES). Thus taking the partial trace with respect to one of the particles (e.g. the one labelled by 1) leaves the the other particle to be, with equal probability, in any state,

$$\rho_2 = \Sigma_n \langle u_n(1)|\psi(1,2)\rangle\langle\psi(1,2)|u_n(1)\rangle = \frac{\mathbb{I}_2}{d}. \tag{2}$$

Two bases $\mathcal{B}_1, \mathcal{B}_2$ are said to be mutually unbiased (MUB) when the decomposition of a basis vector of one basis, $\mathcal{B}_1$ in terms of the other's base vectors contains, with equal amplitude, all the base vectors of $\mathcal{B}_2$. Our aim is to show that such states have a direct interconnection: we establish a relation between the so called d dimensional set of mutually unbiased bases (MUB) [2, 3] and maximally entangled states. This is done via novel MUB that spans the two d dimensional (two) particles Hilbert spaces with collective coordinates bases that we term MUCB. Thus the d^2 dimensional two particles Hilbert sapce is spanned by two, d dimensional collective coordinates bases. The two (one for each particle) d+1 MUB that are available for the particles' Hilbert spaces (we consider d=prime) is replaced ($d \neq 2$) by two d+1 dimensional MUB sets that account for the collective degrees of freedom: we have one set of d+1 MUB for the center of mass and one for the relative coordinates degrees of freedom. The product of of these modes of states is shown to be, almost always, maximally entangled states. Before tackling our subject we give, in section II, a brief review of our approach to MUB for both the continuous ($d \to \infty$) and the finite dimensional, d, Hilbert space. Then in the next section, section III, we consider the continuum case. In this case the rationale involved in this paper is clearest, in particular, we introduce here the collective bases formalism and define the MUCB. In section IV we study the d-dimensional case (with d = prime, $\neq 2$) where advantage is taken of the (algebraic) field theory that is applicable here. (The d=2 case is dealt with somewhat differently.) This section contains the definition and use of collective coordinates and their relation to entanglement for finite dimensional Hilbert space. The last section is devoted to some remarks and concluding statements.

MUTUAL UNBIASED BASES (MUB) - BRIEF REVIEW

We briefly summarize some of the MUB features of the continuous, $d \to \infty$, Hilbert space which will be used later. The complete orthonormal eigenfunctions of the quadrature operator,

$$\hat{X}_\theta \equiv cos\theta \ \hat{x} + sin\theta \ \hat{p} = U^\dagger(\theta)\hat{x}U(\theta), \tag{3}$$

with,

$$U(\theta) = e^{i\theta\hat{a}^\dagger\hat{a}}; \ \ \hat{a} \equiv \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}), \ \hat{a}^\dagger \equiv \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}), \tag{4}$$

are labelled by $|x, \theta\rangle$:

$$\hat{X}_\theta |x, \theta\rangle = x |x, \theta\rangle. \tag{5}$$

(As is well known [5] $\hat{a}$, $\hat{a}^\dagger$ are referred to as "creation" and annihilation" operators, respectively.) This (Eq.(3,5) allow us to relate bases of different labels much like time evolution governed by an harmonic oscillators hamiltonian [6]: we may view the basis labelled by θ as "evolved" from one labelled by $\theta = 0$ i.e. for a vector in the x-representation we have,

$$|x, \theta\rangle = U^\dagger(\theta)|x\rangle. \tag{6}$$

We note that the state whose eigenvalue is x in a basis labelled by θ, i.e. $|x, \theta\rangle$, are related to the eigenfunction, $|p\rangle$, of the momentum operator via,

$$|x, \frac{\pi}{2}\rangle = |p\rangle. \tag{7}$$

(In this equation $|p\rangle$ is the eigenfunction of the momentum operator, $\hat{p}$, whose eigenvalue ,p, is numerically equal to x.) Similarly we have,

$$|x, \pm\pi\rangle = |-x\rangle, \tag{8}$$

i.e. evolution by $\pm\pi$ may be viewed as leading to a vector in the same basis (i.e. θ intact) but evolves to a vector whose eigenvalue is of opposite sign. Returning to Eq.(6) we utilize the known analysis of the evolution operator $U(\theta)$ [6] to deduce that, in terms of the eigenfunction of $\hat{Y}_0 = \hat{x}$, viz the x-representation, $|y; \theta\rangle$ is given by,

$$\langle x|y, \theta\rangle = \langle x|U^\dagger(\theta)|y\rangle = \frac{1}{\sqrt{2\pi sin\theta}} e^{-\frac{i}{2sin\theta}\left([x^2 + y^2]cos\theta - 2xy\right)}. \tag{9}$$

These states form a set of MUB each labelled by θ:

$$|\langle x; \theta|y, \theta'\rangle| = \frac{1}{\sqrt{2\pi|sin(\theta - \theta')|}}. \tag{10}$$

Thus the verification of the particle as being in the state of coordinate x in the basis labelled by θ implies that it is equally likely to be in any coordinate state x' in the basis labelled by θ' ($\theta' \neq \theta$). Note however that in the continuum ($d \to \infty$) considered above, the inter-basis scalar product, Eq.(10), retains, in general, their basis labels (θ, θ'). For a finite, d dimensional Hilbert space the scalar inter MUB product is, in absolute value, $\frac{1}{\sqrt{d}}$ and does not contain any information on the base labels [8]. It was shown by Schwinger [11] that complete operator basis (COB) for this problem constitute of $\hat{Z}$ and $\hat{X}$ with,

$$\hat{Z}|n\rangle = \omega^n |n\rangle, \quad \omega = e^{i\frac{2\pi}{d}}, \quad \hat{X}|n\rangle = |n+1\rangle \quad |n+d\rangle = |n\rangle. \tag{11}$$

It was further shown [2, 3, 4, 7] that the maximal number of MUB possible for a d dimensional Hilbert space is d+1. However only for d=prime (or a power of a prime) d+1 such bases are known to exist. (3 such bases are known for all values of d.) For the case of d=prime a general MUB basis is given in terms of the computational basis,[7]

$$|m; b\rangle = \frac{1}{\sqrt{d}} \Sigma_0^{d-1} \omega^{\frac{b}{2}n(n-1)-nm} |n\rangle. \tag{12}$$

These are the eigenfunction of $\hat{X}\hat{Z}^b$, $b = 0, 1, ...d - 1$. Here b may be used to label the basis. (These d bases supplemented with the computational basis form the d+1 MUB , [7].)

THE CONTINUUM, $d \to \infty$, CASE

The generic maximally entangled state is the EPR [10] state,

$$|\xi, \mu\rangle = \frac{1}{\sqrt{2\pi}} \int dx_1 dx_2 \delta\left(\frac{x_1 - x_2}{\sqrt{2}} - \xi\right) e^{i\mu\frac{x_1 + x_2}{\sqrt{2}}} |x_1\rangle |x_2\rangle, \tag{13}$$

($\sqrt{2}$ is introduced for later convenience.) We now consider an alternative means of accounting for the two particles states to which we refer to as the "relative" and "center of mass" coordinates (we assume equal masses for simplicity),

$$\xi = \frac{x_1 - x_2}{\sqrt{2}}; \quad \eta = \frac{x_1 + x_2}{\sqrt{2}}. \tag{14}$$

The corresponding operators, each acting on one of these coordinates, are

$$\hat{\xi} = \frac{\hat{x}_1 - \hat{x}_2}{\sqrt{2}}; \quad \hat{\eta} = \frac{\hat{x}_1 + \hat{x}_2}{\sqrt{2}}, \tag{15}$$

with,

$$\hat{\xi}|\xi\rangle = \xi|\xi\rangle; \quad \hat{\eta}|\eta\rangle = \eta|\eta\rangle. \tag{16}$$

Using relations of the type,

$$\langle x_1 x_2 | \hat{\xi} | \xi\eta\rangle = \xi\langle x_1 x_2 | \xi\eta\rangle = \langle x_1 x_2 | \frac{\hat{x}_1 - \hat{x}_2}{\sqrt{2}} | \xi\eta\rangle = \frac{x_1 - x_2}{\sqrt{2}} \langle x_1 x_2 | \xi\eta\rangle, \tag{17}$$

One may show that,

$$\langle x_1 x_2 | \xi\eta\rangle = \delta\big(\xi - \frac{x_1 - x_2}{\sqrt{2}}\big)\delta\big(\eta - \frac{x_1 + x_2}{\sqrt{2}}\big). \tag{18}$$

We note that $\hat{x}_1, \hat{p}_1$ form a complete operator basis (COB) for the first particle Hilbert space (we do not involve spin) and similarly $\hat{x}_2, \hat{p}_2$ for the second particle, i.e.,

$$\begin{aligned} [\hat{x}_1, \hat{p}_1] &= [\hat{x}_2, \hat{p}_2] = i, \\ [\hat{x}_1, \hat{p}_2] &= [\hat{x}_2, \hat{p}_1] = [\hat{x}_2, \hat{x}_1] = [\hat{p}_2, \hat{p}_1] = 0, \end{aligned} \tag{19}$$

thus we have that the two pairs of operators form a COB for the combined (d^2 dimensional) Hilbert space. Defining,

$$\hat{\nu} \equiv \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{2}}, \quad \hat{\mu} \equiv \frac{\hat{p}_1 + \hat{p}_2}{\sqrt{2}}, \tag{20}$$

we have

$$\begin{aligned} \left[\hat{\xi}, \hat{\nu}\right] &= [\hat{\eta}, \hat{\mu}] = i, \\ \left[\hat{\xi}, \hat{\mu}\right] &= \left[\hat{\xi}, \hat{\eta}\right] = [\hat{\eta}, \hat{\nu}] = [\hat{\mu}, \hat{\nu}] = 0. \end{aligned} \tag{21}$$

These (viz: $\hat{\xi}, \hat{\nu}, \hat{\eta}, \hat{\mu}$) form an alternative COB for the (combined) Hilbert space with $\hat{\xi}, \hat{\nu}$ spanning the relative coordinates space while $\hat{\eta}, \hat{\mu}$ the "center of mass" one. By analogy with the single particle state analysis we now define "creation" and "annihilation" operators for the collective degrees of freedom:

$$\begin{aligned} \hat{A} &= \frac{1}{\sqrt{2}}(\hat{\xi} + i\hat{\nu}), \quad \hat{A}^\dagger = \frac{1}{\sqrt{2}}(\hat{\xi} - i\hat{\nu}), \\ \hat{B} &= \frac{1}{\sqrt{2}}(\hat{\eta} + i\hat{\mu}), \quad \hat{B}^\dagger = \frac{1}{\sqrt{2}}(\hat{\eta} - i\hat{\mu}), \end{aligned} \tag{22}$$

these abide by the commutation relations

$$\left[\hat{A}, \hat{A}^\dagger\right] = \left[\hat{B}, \hat{B}^\dagger\right] = 1, \tag{23}$$

with all other commutators vanishing, and the "evolution" (Eq.(6)) operators are,

$$V_A(\theta) = e^{i\theta \hat{A}^\dagger \hat{A}}; \quad V_B(\theta) = e^{i\theta \hat{B}^\dagger \hat{B}}. \tag{24}$$

These operators are (as we shall see shortly) our entangling operators: each (pair) act on different "collective" coordinate. We note that for $\theta = \theta'$ and only in this case,

$$V_A^\dagger(\theta)V_B^\dagger(\theta) = U_1^\dagger(\theta)U_2^\dagger(\theta), \tag{25}$$

i.e. in this case a simple relation exists between the particles' operators and the collective ones. The results of section II, Eq.(6), now read,

$$|\xi, \theta\rangle = V_A^\dagger(\theta)|\xi\rangle,$$
$$|\eta, \theta'\rangle = V_B^\dagger(\theta')|\eta\rangle. \tag{26}$$

The commutation relation, Eq.(21), implies that the basis $|\eta\rangle$, the eigenbasis of $\hat{\eta}$ (i.e. $V_B^\dagger(0)|\eta\rangle$), and the basis $|\mu\rangle$, the eigenstates of $\hat{\mu}$, (i.e. the states $V_B^\dagger(\frac{\pi}{2}))|\eta\rangle$, are MUB with,

$$\langle \eta|\mu\rangle = \frac{1}{\sqrt{2\pi}}e^{i\eta\mu}, \tag{27}$$

With similar expression for $\langle \xi|\nu\rangle$. Note that, in our approach, these follow from the equations that corresponds to Eq.(9). We have then that the maximally entangled state (the EPR state)

$$|\xi\rangle|\mu\rangle$$

is a product state in the collective variables. It is natural now to consider mutual unbiased collective bases (MUCB) labelled, likewise, with θ: The relative coordinates bases $V_A^\dagger(\theta)|\xi\rangle$ is one such MUCB. The center of mass $V_B^\dagger(\theta)|\eta\rangle$ is another. We now formulate our link between MUB and (maximal) entanglement thus consider the (product) two particle state $|x_1\rangle|x_2\rangle$. It may be written in terms of a product state in the "collective" coordinates: (When clarity requires we shall mark henceforth the eigenstates of the collective operators with double angular signs, $\rangle\rangle$,.)

$$|x_1\rangle|x_2\rangle = \int d\xi' d\eta' \langle \xi', \eta'|x_1, x_2\rangle|\xi'\rangle\eta'\rangle = |\xi = \frac{x_1 - x_2}{\sqrt{2}}\rangle\rangle|\eta = \frac{x_1 + x_2}{\sqrt{2}}\rangle\rangle. \tag{28}$$

We now assert that replacing the basis $|\eta\rangle$ by any of the MUB bases,

$$|\eta\rangle \rightarrow |\eta, \theta\rangle = V_B^\dagger(\theta)|\eta\rangle, \ \theta \neq 0,$$

give a maximally entangled state: $|\xi\rangle|\eta, \theta\rangle$. (The EPR state, $|\xi, \mu\rangle$ is the special case of $V_B^\dagger(\frac{\pi}{2})$.) The proof is most informative with the state $|\xi\rangle|\mu, \theta\rangle$: (Note: $|\mu, \theta\rangle = V_B(\theta)|\mu\rangle = V_B(\theta + \frac{\pi}{2})|\eta\rangle$.)

$$|\xi\rangle|\mu, \theta\rangle = \int dx_1 dx_2 |x_1, x_2\rangle\langle x_1, x_2| \int d\eta d\bar{\eta} \ |\xi, \eta\rangle\langle \eta|\bar{\eta}, \theta\rangle d\bar{\eta}\langle \bar{\eta}, \theta|\mu, \theta\rangle$$
$$= \frac{\sqrt{2}}{2\pi \cos\theta}e^{\frac{i\mu}{2\cos\theta}\left(2\xi - \mu\sin\theta\right)} \int dx e^{\frac{\sqrt{2}ix\mu}{\cos\theta}}|x\rangle|x - \sqrt{2}\xi\rangle. \tag{29}$$

The various matrix elements are given by,

$$\langle x_1|\xi, \eta\rangle = \delta\left(x_1 - \frac{\eta + \xi}{\sqrt{2}}\right)\left|x_2 = \frac{\eta - \xi}{\sqrt{2}}\right\rangle,$$
$$\langle \eta|\bar{\eta}, \theta\rangle = \frac{1}{\sqrt{2\pi|sin\theta|}}e^{-\frac{i}{2sin\theta}\left[(\eta^2 + \bar{\eta}^2)cos\theta - 2\bar{\eta}\eta\right]},$$
$$\langle \bar{\eta}, \theta|\mu, \theta\rangle = \frac{1}{\sqrt{2\pi}}e^{i\bar{\eta}\mu}. \tag{30}$$

The state, Eq.(29), upon proper normalization, is the maximally entangled EPR state, as claimed (cf. Appendix B): it involves, with equal probability, all the vectors of the x representation. It follows by inspection that this remain valid to all states (exceptions are specific angles that are specified below) build with MUCB,

$$|\xi, \theta\rangle|\eta, \theta'\rangle. \tag{31}$$

We summarize our consideration thus far as follows: Consider two pairs of operators (we assume that these two form a COB) pertaining to two Hilbert spaces. Each pair is made up of *non commuting* operators, e.g. $\hat{x}_1, \hat{p}_1$ and $\hat{x}_2, \hat{p}_2$. Now form two *commuting* pairs of operators with these operators as their constituents, e.g. $\hat{R}_A(0) = \hat{x}_1 - \hat{x}_2$

and $\hat{R}_B(\frac{\pi}{2}) = \hat{p}_1 + \hat{p}_2$: the common eigenfunction of $\hat{R}_A(0)$ and $\hat{R}_B(\frac{\pi}{2})$ is, necessarily, an entangled state. This was generalized via the consideration of the common eigenfunction of the commuting operators

$$
\begin{aligned}
R_A(\theta) &\equiv V_A^\dagger(\theta)\hat{\xi}V_A(\theta) = cos\theta\hat{\xi} + sin\theta\hat{\nu}, \\
R_B(\theta') &\equiv V_B^\dagger(\theta')\hat{\eta}V_B(\theta') = cos\theta'\hat{\eta} + sin\theta'\hat{\mu}.
\end{aligned}
\tag{32}
$$

These commute for all θ, θ' and thus have common eigenfunctions. For $\theta = \theta'$ *and* $\theta = \theta' \pm \pi$ and only for these values, the common eigenfuction is a product state (in these cases the constituents commute, e.g. $\hat{x}_1 - \hat{x}_2$ and $\hat{x}_1 + \hat{x}_2$). This is shown in Appendix C. For all other θ, θ' the common eigenfunction is an entangled state. (Moreover, these states are maximally entangled states. The proof is outlined in Appendix B.) The definition of the "collective" coordinates is such as to assure the decoupling of the combined Hilbert space to two independent subspaces whose constituent (pairs) operators commute (e.g. $\hat{x}_1 - \hat{x}_2$ and $\hat{x}_1 + \hat{x}_2$) much as it (the Hilbert space) was decoupled with the individual particles operators.

FINITE DIMENSIONAL ANALYSIS - COLLECTIVE COORDINATES

We now turn to the more intriguing cases of d dimensional Hilbert spaces. We confine our study to (two) d-dimensional spaces with d a prime ($\neq 2$). The indices are elements of an algebraic field of order d. The computational, two particle, basis states

$$
|n\rangle_1 |m\rangle_2 \ \ n, m = 0, 1, ..d-1.
$$

spans the space. A COB (complete operator basis) is defined via ($i = 1, 2$,

$$
\begin{aligned}
Z_i|n\rangle_i &= \omega^{n_i}|n\rangle_i, \ \ \omega = e^{i\frac{2\pi}{d}} \\
X_i|n\rangle_i &= |n+1\rangle_i,
\end{aligned}
\tag{33}
$$

We now define our collective coordinate operators via,

$$
\bar{Z}_1 \equiv Z_1^{\frac{1}{2}} Z_2^{-\frac{1}{2}}; \ \bar{Z}_2 \equiv Z_1^{\frac{1}{2}} Z_2^{\frac{1}{2}}.
\tag{34}
$$

(We remind the reader that the exponent value of $\frac{1}{2}$ is a field number such that twice its value is 1 mode[d], e.g. for d=7, $\frac{1}{2} = 4$.) Eq.(34) implies that,

$$
Z_1 = \bar{Z}_1 \bar{Z}_2, \ \ Z_2 = \bar{Z}_1^{-1} \bar{Z}_2.
\tag{35}
$$

The spectrum of $\bar{Z}_i$ is $\omega^{\bar{n}}, \bar{n} = 0, 1, ..d-1$ since we have that $\bar{Z}_i^d = 1$ and we consider the bases that diagonalize $\bar{Z}_i$:

$$
\bar{Z}_i|\bar{n}_i\rangle = \omega^{\bar{n}_i}|\bar{n}_i\rangle.
\tag{36}
$$

To obtain the transformation function $\langle n_1, n_2 | \bar{n}_1, \bar{n}_2 \rangle$ we evaluate $\langle n_1, n_2 | A | \bar{n}_1, \bar{n}_2 \rangle$ with A equals $Z_1, Z_2, \bar{Z}_1, \bar{Z}_2$ in succession. e.g. for $A = Z_1$,

$$
\langle n_1, n_2 | Z_1 | \bar{n}_1, \bar{n}_2 \rangle = \omega^{n_1} \langle n_1, n_2 | \bar{n}_1, \bar{n}_2 \rangle = \omega^{\bar{n}_1 + \bar{n}_2} \langle n_1, n_2 | \bar{n}_1, \bar{n}_2 \rangle.
\tag{37}
$$

These give us the following relations (all equations are modular: mode[d]),

$$
\begin{aligned}
n_1 &= \bar{n}_1 + \bar{n}_2; \ \ n_2 = -\bar{n}_1 + \bar{n}_2, \\
\bar{n}_1 &= \frac{n_1}{2} - \frac{n_2}{2}; \ \ \bar{n}_2 = \frac{n_1}{2} + \frac{n_2}{2}.
\end{aligned}
\tag{38}
$$

Whence we deduce,

$$
\langle n_1, n_2 | \bar{n}_1, \bar{n}_2 \rangle = \delta_{n_1, \bar{n}_1 + \bar{n}_2} \delta_{n_2, \bar{n}_1 - \bar{n}_2}.
\tag{39}
$$

In a similar fashion we now define,

$$
\bar{X}_1 \equiv X_1 X_2^{-1}, \ \ \bar{X}_2 \equiv X_1 X_2 \ \rightarrow X_1 = \bar{X}_1^{1/2} \bar{X}_2^{1/2}, \ X_2 = \bar{X}_1^{-1/2} \bar{X}_2^{1/2}.
\tag{40}
$$

These entail,

$$\bar{X}_i \bar{Z}_i = \omega \bar{Z}_i \bar{X}_i, \ i = 1, 2$$
$$\bar{X}_i \bar{Z}_j = \bar{Z}_j \bar{X}_i, \ i \neq j. \tag{41}$$

Thence,

$$\bar{X}_i | \bar{n}_i \rangle = | \bar{n}_i + 1 \rangle, \quad i = 1, 2 \tag{42}$$

and, denoting the eigenfunction of the barred operators (i.e. the collective coordinates) with double angular sign we have that

$$\bar{X}_1 | n_1, n_2 \rangle = \bar{X}_1 | \frac{n_1 - n_2}{2}, \frac{n_1 + n_2}{2} \rangle\rangle = | \frac{n_1 - n_2}{2} + 1, \frac{n_1 + n_2}{2} \rangle\rangle$$
$$\bar{X}_2 | n_1, n_2 \rangle = \bar{X}_2 | \frac{n_1 - n_2}{2}, \frac{n_1 + n_2}{2} \rangle\rangle = | \frac{n_1 - n_2}{2}, \frac{n_1 + n_2}{2} + 1 \rangle\rangle. \tag{43}$$

Recalling, Eq.(10), the set of MUB associated with $|\bar{n}_2\rangle\rangle$, viz $|\bar{n}_2, b\rangle\rangle$ (with $b = 0, 1..d-1$):

$$|\bar{n}_2, b\rangle = \frac{1}{\sqrt{d}} \Sigma_{\bar{n}} \omega^{\frac{b}{2} \bar{n}(\bar{n}+1) - \bar{n}\bar{n}_2} |\bar{n}\rangle\rangle. \tag{44}$$

This state is an eigenfunction of $\bar{X}_2 \bar{Z}_2^b$, cf Eq. (). Our association of maximally entangled states with MUB amounts to the following. Given a product state. We write it as a product state of the collective coordinates, e.g.

$$|n_1\rangle |n_2\rangle = |\bar{n}_1\rangle\rangle |\bar{n}_2\rangle\rangle, \ n_1 = \bar{n}_1 + \bar{n}_2; \ n_2 = \bar{n}_2 - \bar{n}_1. \tag{45}$$

Now replace one of these (collective coordinates states) by a state (any one of which) belonging to its MUB set, e.g.

$$|\bar{n}_1\rangle\rangle |\bar{n}_2\rangle\rangle \rightarrow |\bar{n}_1\rangle\rangle |\bar{n}_2, b\rangle\rangle, \quad b = 1, 2..d-1. \tag{46}$$

The resultant state is a maximally entangled state. We prove it for a representative example by showing that measuring in such state Z_1 that yield the value n_1 leaves the state an eigenstate of Z_2 with a specific eigenvalue. To this end we consider the projection of the state $\langle n_1 |$ on the representative state . Somewhat lengthy calculation yields,

$$\langle n_1 | \bar{n}_1 \rangle\rangle |\bar{n}_2, b\rangle\rangle = \frac{1}{\sqrt{d}} |n_2 = -2\bar{n}_1 + n_1\rangle \omega^{\frac{b}{2}(n_1 - \bar{n}_1)(n_1 - \bar{n}_1 - 1) - \bar{n}_2(n_1 - \bar{n}_1)}. \tag{47}$$

Here the state $|n_2 = -2\bar{n}_1 + n_1\rangle$ is an eigenstate of Z_2 proving our point.

We discuss now the finite dimensional Hilbert space in a manner that stresses its analogy with the $d \rightarrow \infty$ case considered above: Given two, each d-dimensional, Hilbert spaces and each pertaining to one of two particles (systems) bases. The combined, d^2-dimensional space is conveniently spanned by a basis made of product of computational bases, $|n_1\rangle |n_2\rangle$; $n_i = 0, 1, ...d-1$. Each of the computational basis may be replaced by any of the d other available MUB bases (recall that we limit ourselves to d=prime where d+1 MUB are available [7]). Each MUB basis is associated [7] with a unitary operator, $X_i Z_i^b$, $b = 0, 1, ..d-1$ (these supplemented by Z_i account for the d+1 MUB). We have shown above that the combined Hilbert space may be accounted for by what we termed collective coordinates computational bases: $|\bar{n}_1\rangle |\bar{n}_2\rangle$, $\bar{n}_i = 0, 1, ...d-1$. (Here $|\bar{n}_1\rangle$ relates to the "relative" while $|\bar{n}_2\rangle$ to the "center of mass" coordinate.) These were defined such that

$$|n_1\rangle |n_2\rangle = |\bar{n}_1\rangle\rangle |\bar{n}_2\rangle\rangle.$$

We then noted that, in analogy with the $R_i(\theta)$ of the $d \rightarrow \infty$ case each $|\bar{n}_i\rangle\rangle$ may be replaced by any of the d+1 MUB of the collective coordinates. These are associated with $\bar{X}_i \bar{Z}_i^b$, $b = 0, 1, ...d-1$. We now have the space spanned by $|\bar{n}_1, b_1\rangle\rangle |\bar{n}_2, b_2\rangle\rangle$. These except for "isolated" combination are maximally entangled states (cf. Appendix A). The isolated values are the $b_1 = b_2$ cases and the bases associated with $\bar{X}_1 \bar{Z}_1^b$ and $\bar{X}_2^{-1} \bar{Z}_2^{-b}$ - the eigenstates of which are product states.

Now while in the finite dimensional case the set of d+1 MUB states can be constructed only for d a prime (or a power of a prime - which is not studied here) no such limit holds for the continuous case. The intriguing price being that in this $(d \rightarrow \infty)$ case the definition of the MUB states involves a basis dependent normalization. We have considered the

cases with d=prime. The case d=2 need special treatment because, in this case, +1=-1 [mode 2] (indeed 2=0 [mode 2]) hence the "center of mass" and "relative" coordinates are indistinguishable. Here the operator vantage point may be used to interpret the known results [11]. The operators vantage point involves the following: given two systems α, β. Consider two non commuting operators pertaining to α : A, A' and correspondingly two non-commuting operators B and B' that belong to β. our scheme was to construct a common eigenfunction for AB and A'B' with (which we assume) $[AB, A'B'] = 0$. This common eigenfunctions are maximally entangled. This is trivially accomplished: e.g. consider $(\alpha, \beta \to 1, 2 \; \sigma_{x1}\sigma_{x2} \; with \; \sigma_{z1}\sigma_{z2}$, and $\sigma_{x1}\sigma_{x2} \; with \; \sigma_1\sigma_{y2}$. Their common eigenfunctions are the well known Bell states [9].

CONCLUDING REMARKS

An association of maximally entangled states for two particles, each of dimensionality d, with mutually unbiased bases (MUB) of d dimensional Hilbert space inclusive of the continuous $(d \to \infty)$ cases were established. The analysis is based on the alternative forms for the two particle state: product of computational based states, and a product of the state given in terms of collective coordinates (dubbed center of mass and relative). A formalism allowing such an alternative accounting for the states was developed for d a prime $(\neq 2)$ which applies the finite, d $(\neq 2)$, dimensional cases where the maximally allowed MUB $(d+1)$ is known to be available. Based on the alternative ways of writing the two particle states we defined and demonstrated the use of mutually unbiased collective bases (MUCB). The latter is generated by noting that replacing one of the states in the collective coordinates product state with any of its MUCB states realizes a maximal entangled state. Such state is, by construction, made of eigenfunctions of commuting pairs of two particles operators with the single particle operators in the different pair non commuting. Thus we shown that maximally entangled states both in the continuum and some finite dimension Hilbert spaces may be viewed as product states in collective variables and have demonstrated the intimate connection between entanglement and operator non commutativity (i.e. the uncertainty principle).

Appendix A: Maximally Entangled State

We prove here that the state $|\xi\rangle|\eta, \frac{\pi}{2}\rangle$ is a maximally entangled state. (Note $|\eta, \theta + \frac{\pi}{2}\rangle = |\mu, \theta\rangle$). This can be seen directly by calculating the x representation of the state and noting that it is of the same form of the EPR state, i.e. its Schmidt decomposition contains all the states paired with coefficients of equal magnitude [1, 12]:

$$
\begin{aligned}
|\xi\rangle|\mu, \theta\rangle &= \int dx_1 dx_2 |x_1, x_2\rangle\langle x_1, x_2| \int d\eta d\bar{\eta} \, |\xi, \eta\rangle\langle\eta|\bar{\eta}, \theta\rangle d\bar{\eta}\langle\bar{\eta}, \theta|\mu, \theta\rangle \\
&= \frac{\sqrt{2}}{2\pi \cos\theta} e^{\frac{i\mu}{2\cos\theta}\left(2\xi - \mu\sin\theta\right)} \int dx e^{\frac{\sqrt{2}ix\mu}{\cos\theta}} |x\rangle|x - \sqrt{2}\xi\rangle.
\end{aligned}
\tag{48}
$$

This is a maximally entangled state for $0 \le \theta < \frac{\pi}{2}$. Now considering the state for $\theta = \frac{\pi}{2}$ we have (c.f., Eq.(7,8)

$$
\langle x_1, x_2|\xi\rangle|\mu, \frac{\pi}{2}\rangle = \langle x_1, x_2|\xi, -\eta\rangle = \delta\left(x_1 - \frac{\xi - \eta}{\sqrt{2}}\right)\left(x_2 + \frac{\xi + \eta}{\sqrt{2}}\right),
\tag{49}
$$

i.e. at $\theta = \frac{\pi}{2}$ the state is a product state. We interpret this to mean that entanglement is not analytic [8].

Appendix B: Maximal entanglement of the state $|\xi, \theta; \eta, \theta'\rangle$

We now prove that the state $|\xi, \theta, \eta, \theta'\rangle$ is a maximally entangled state for all θ, θ' (except for isolated points:$\theta = \theta' \pm \pi$, at these points the state is a product state). We note that

$$
\hat{A}^\dagger\hat{A} + \hat{B}^\dagger\hat{B} = \hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2.
\tag{50}
$$

Hence, cf. Eq. (25), here the numerical subscripts refers to the particles,

$$
V_A^\dagger(\theta)V_B^\dagger(\theta) = U_1^\dagger(\theta)U_2^\dagger(\theta).
\tag{51}
$$

Assuming, without loss of generality that $\theta' > \theta$ (when they are equal the state is a product state), we may thus write $(\Delta = \theta' - \theta)$,

$$|\xi,\theta;\eta,\theta'\rangle = \int d\bar{\eta}\left|x_1 = \frac{\bar{\eta}+\xi}{\sqrt{2}},\theta\right\rangle\left|x_2 = \frac{\bar{\eta}-\xi}{\sqrt{2}},\theta\right\rangle\frac{1}{\sqrt{2\pi|sin\Delta|}}e^{-\frac{i}{2sin\Delta}\left[(\eta^2+\bar{\eta}^2)cos\Delta-2\bar{\eta}\eta\right]}. \tag{52}$$

Here the vectors $(|x_i\rangle)$ are the single particle eigenvectors of $U^\dagger(\theta)\hat{x}_iU(\theta)$. Now our proof that the state $|\xi,\theta;\eta,\theta'\rangle$ is a maximally entangled state is attained via "measuring" the position of the first particle (in the basis labelled by θ), i.e. calculating the projection $\langle x_1',\theta|\xi,\theta;\eta,\theta'\rangle$, and showing that the resultant state is the second particle in a definite (up to a phase factor) one particle state, $|y_2,\bar{\theta}\rangle$ with y_2 linearly related to x_1'. Thus $(x' = x_1')$:

$$\langle x_1'|\xi,\theta;\eta,\theta'\rangle = \frac{1}{\sqrt{\pi|sin\Delta|}}e^{\frac{i}{2sin\Delta}\left[2\xi\eta+(\xi^2+\eta^2)cos\Delta\right]}e^{\frac{i}{sin\Delta}\left[(x'^2-\sqrt{2}x'\xi)cos\Delta-\sqrt{2}x'\eta\right]}|x'-\sqrt{2}\xi\rangle. \tag{53}$$

QED

Appendix C: Angular labels for product states

The proof that $|\xi,\theta\rangle|\eta,\theta'\rangle$ are product states for $\theta = \theta'$ *and* $\theta = \theta'\pm\pi$ utilizes the following preliminary observations:
a. $V_A^\dagger(\pm\pi)|\xi\rangle = |-\xi\rangle$; $V_B^\dagger(\pm\pi)|\eta\rangle = |-\eta\rangle$ i.e. "evolution" by $\pm\pi$ may be viewed as leaving the basis unchanged but "evolves" to a state whose eigenvalue is of opposite sign. See Eq. (8).
The states $|\xi\rangle|\pm\eta\rangle$, $|\pm\xi\rangle|\eta\rangle$ are product states: e.g.

$$|\xi\rangle|-\eta\rangle = \int dx_1dx_2|x_1\rangle_1|x_2\rangle\langle x_1|\langle x|\xi\rangle|-\eta\rangle =$$

$$\int dx_1dx_2|x_1\rangle|x_2\rangle\delta\left(\xi-\frac{x_1-x_2}{\sqrt{2}}\right)\delta\left(-\eta-\frac{x_1+x_2}{\sqrt{2}}\right) = \left|\frac{\xi-\eta}{\sqrt{2}}\right\rangle\left|-\frac{\eta+\xi}{\sqrt{2}}\right\rangle.$$

QED.
These observations imply that, e.g.,

$$|\xi,\theta\rangle|\eta,\theta+\pi\rangle = V_A^\dagger(\theta)V_B^\dagger(\theta+\pi)|\xi\rangle|\eta\rangle =$$

$$U_1^\dagger(\theta)U_2^\dagger|\xi\rangle|-\eta\rangle = \left|\frac{\xi-\eta}{\sqrt{2}};\theta\right\rangle\left|-\frac{\eta+\xi}{\sqrt{2}};\theta\right\rangle.$$

With similar results for $|\pm\xi\rangle|\eta\rangle$. These are are product states each involves a distinct particle.

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